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Canonical Products and the Weights $\exp(-|x|^\alpha)$, $\alpha > 1$, with Applications

A. L. LEVIN*

Department of Mathematics, Everyman's University, P.O. Box 39328, Ramat Aviv, Tel Aviv, Israel

AND

D. S. LUBINSKY

National Research Institute for Mathematical Sciences of the CSIR, P.O. Box 395, Pretoria 0001, Republic of South Africa

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Let $\alpha > 1$. For each positive integer n, a polynomial $S_n(x)$ of degree $\leqslant n$ is constructed such that $S_n(x) \sim \exp(-|x|^\alpha)$, $|x| \leqslant Cn^{1/\alpha}$, where $C > 0$ is independent of n. These polynomials enable one to estimate Christoffel functions and prove L_p Markov–Bernstein inequalities for all $0 < p \leqslant \infty$, and for all the weights $\exp(-|x|^\alpha)$, $\alpha > 1$. In particular, the gap $1 < \alpha < 2$ in Feud's approximation theory can be filled, and one can prove L_p Markov–Bernstein inequalities for $0 < p < 1$.

1. INTRODUCTION

The early papers of Freud [6, 7] and Nevai [21, 22] on weighted approximation for exponential weights dealt primarily with weights such as $W_{2k}(x) = \exp(-x^{2k})$, where k is a positive integer, The reason for this was that the $(n+1)$th partial sum, $S_n(x)$ say, of the Maclaurin series for $W_{2k}(x)$ satisfies

$$S_n(x) \sim W_{2k}(x), \qquad |x| \leqslant Cn^{1/2k}, \tag{1.1}$$

where $C > 0$ is independent of n and x. Here $\sim$ denotes that the ratio of S_n and W_{2k} is bounded above and below by positive constants independent

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of n and x. These polynomials facilitated estimation of Christoffel functions and proofs of Markov–Bernstein inequalities.

Subsequently, Freud [8–10] realized that, in order to provide upper and lower bounds for Christoffel functions, it suffices to construct polynomials that equal the weight at one point and that approximate the weight on one side only, rather than satisfy (1.1). This, and other ideas, enables Freud to prove weighted Markov–Bernstein inequalities in $L_p(1 \leqslant p \leqslant \infty)$, and to develop a theory of weighted approximation for a large class of weights, which included $W_\alpha(x) = \exp(-|x|^\alpha)$, $\alpha \geqslant 2$. A partial theory of approximation for $W_1(x)$ was developed by Freud, Giroux, and Rahman [12].

One gap in Freud's theory was that it did not treat the weights $W_\alpha(x)$, $1 < \alpha < 2$: The methods of [6–10, 21, 22] did not yield lower bounds for the Christoffel functions to match the upper bounds in [9]. The partial lower bounds obtained by one of us [17, Theorem 3.6(ii)] and Mhaskar and Saff [20, Theorem 6.5(b)] do not really fill this gap. See [17] for further results and references on Christoffel functions, and Mhaskar [19] and Nevai [25] for more general surveys.

A futher gap in Freud's theory is that his method of proving L_p Markov–Bernstein inequalities does not work for $0 < p < 1$. Using the polynomials $\{S_n(x)\}$ that satisfy (1.1), Bonan [3] and Bonan and Nevai [4] filled this gap for weights such as $|x|^\beta \exp(-x^2)$. In this paper the gaps $1 < \alpha < 2$ and $0 < p < 1$, $\alpha > 1$ will be filled. One of the main results of this paper is

THEOREM 1.1. *Let $\alpha > 1$. There exist even polynomials $S_n(x)$, $n = 1, 2, \ldots$, such that*

(i) *$S_n(x)$ has degree at most n,*

(ii) $$S_n(x) \sim W_\alpha(x), \quad |x| \leqslant C_1 n^{1/\alpha}, \tag{1.2}$$

(iii) $$|S_n'(x)| \leqslant C_2 |x|^{\alpha-1} W_\alpha(x), \quad |x| \leqslant C_1 n^{1/\alpha}, \tag{1.3}$$

and in particular

$$|S_n'(x)| \leqslant C_3 n^{1-1/\alpha} W_\alpha(x), \qquad |x| \leqslant C_1 n^{1/\alpha}.$$

Here the constants C_1, C_2, and C_3 are independent of n and x.

It will be shown after Theorem 7.4 that assertions (i) and (ii) of Theorem 1.1 cannot hold for any $\alpha \in (0, 1]$. For $\alpha \geqslant 2$, one can replace the $\leqslant$ in (1.3) by $\sim$, at least for $|x| \geqslant 1$, but the proof of this will be omitted. One easy consequence of Theorem 1.1 is

COROLLARY 1.2. (Estimates of Christoffel functions). *Let* $\alpha > 1$. *For* $n = 1, 2, 3, \ldots,$ *and* $x \in \mathbb{R}$, *let*

$$\lambda_n(W_\alpha^2, x) = \inf \int_{-\infty}^{\infty} (PW_\alpha)^2(u)\, du / P^2(x),$$

where the infimum is taken over all polynomials $P(x)$ *of degree at most* $n-1$. *There exist positive constants* C_1 *and* C_2 *independent of* n *and* x, *such that*

$$\text{(i)} \quad \lambda_n(W_\alpha^2; x) \sim n^{1/\alpha - 1} W_\alpha^2(x), \; |x| \leqslant C_1 n^{1/\alpha}, \tag{1.4}$$

$$\text{(ii)} \quad \lambda_n(W_\alpha^2; x) \geqslant C_2 n^{1/\alpha - 1} W_\alpha^2(x), \; x \in \mathbb{R}. \tag{1.5}$$

Corollary 1.2 is a special case of Theorem 7.4. A further straightforward consequence of Theorem 1.1 is the "local" Markov–Bernstein inequality in Theorem 7.3. The most important special case is

COROLLARY 1.3. (Markov–Bernstein Inequality). *Let* $\alpha > 1$ *and* $0 < p \leqslant \infty$. *For* $n = 1, 2, \ldots,$ *and all polynomials* P *of degree at most* n,

$$\|P' W_\alpha\|_{L_p(\mathbb{R})} \leqslant C n^{1 - 1/\alpha} \|PW_\alpha\|_{L_p(\mathbb{R})},$$

where C *is independent of* n *and* P.

To construct the polynomials of Theorem 1.1, we consider entire functions that are canonical products of Weierstrass primary factors with only negative real zeros. The general asymptotic results for canonical products in Boas [2] and Levin [14] are not sufficiently precise for our purposes, but those in Abi-Khuzam [1] are. However, our approach is different from that in [1], since we wish to treat more general weights in a subsequent paper, and thus prove some of the preliminary results in a general form.

Unfortunately, in considering general weights $W(x) = \exp(-Q(x))$, one has to use the results in [18], and one has to treat separately the cases where $Q(t)$ grows slower, or at least as fast as t^2, as $|t| \to \infty$. There is a further problem when $Q(t)$ grows slightly slower than t^2 or t^4, as $|t| \to \infty$. Hence the decision to treat only the weights $W_\alpha(x)$, $\alpha > 1$, in this paper.

The paper is organized as follows: In Section 2, we present our notation. In Section 3, some preliminary lemmas are established, and in Section 4, a "remainder" term is approximated by polynomials. In Section 5, we use the partial sums of Maclaurin series to approximate certain entire functions by polynomials, and in Section 6, the proof of Theorem 1.1 is completed. Section 7 contains the proofs of Corollaries 1.2 and 1.3, as well as several further results.

2. Notation

Given $\alpha > 0$, we let

$$W_\alpha(x) = \exp(-|x|^\alpha), \qquad x \in \mathbb{R}.$$

Throughout, $p_n(W_\alpha^2; x)$, $n = 0, 1, 2, \ldots$, denote the orthonormal polynomials for W_α^2, satisfying

$$\int_{-\infty}^{\infty} p_n(W_\alpha^2; x)\, p_m(W_\alpha^2; x)\, W_\alpha^2(x)\, dx = \delta_{mn}, \qquad m, n = 0, 1, 2, \ldots.$$

Further, for $n = 1, 2, 3, \ldots$ and all $x \in \mathbb{R}$,

$$\lambda_n(W_\alpha^2, x) = \inf \int_{-\infty}^{\infty} (PW_\alpha)^2(u)\, du / P^2(x),$$

where the infimum is taken over all $P \in \mathcal{P}_{n-1}$, the class of real polynomials of degree at most $n-1$. Given $0 < p \leqslant \infty$, $\|\cdot\|_{L_p(\mathbb{R})}$ denotes the usual L_p norm on $\mathbb{R}$. As in [17], for $0 < p \leqslant \infty$, and non-negative integers j, we define

$$\lambda_{n,p}(W_\alpha, j, x) = \inf_{\mathcal{P}_{n-1}} \|PW_\alpha\|_{L_p(\mathbb{R})} / |P^{(j)}(x)|, \qquad x \in \mathbb{R}, \tag{2.1}$$

$n = j+1, j+2, \ldots$. Note that, as in [17],

$$\{\lambda_{n,2}(W_\alpha, j, x)\}^{-2} = \sum_{k=0}^{n-1} \{p_k^{(j)}(W_\alpha^2; x)\}^2, \tag{2.2}$$

and

$$\{\lambda_{n,2}(W_\alpha, 0, x)\}^2 = \lambda_n(W_\alpha^2, x). \tag{2.3}$$

Throughout, $C, C_1, C_2, \ldots$ denote positive constants independent of n and x. Different occurences of the same symbol do not necessarily denote the same constant. When stating inequalities for polynomials P of degree at most n, the constants will be independent of P, n, and x. To denote dependence of constants C on parameters $\alpha, p, \ldots$, we write $C = C(\alpha, p)$, and so on.

The usual symbols $\sim$, o, and 0 will be used to compare functions and sequences. Thus, $f(x) \sim g(x)$ if for some C_1 and C_2, $C_1 \leqslant f(x)/g(x) \leqslant C_2$ for all x considered. Given a non-negative integer l, the Weierstrass primary factor of order l is

$$E(z, l) = \begin{cases} (1-z) & l = 0 \\ (1-z)\exp(z + z^2/2 + \cdots + z^l/l), & l > 0. \end{cases}$$

Finally, for any real x, $[x]$ denotes the largest integer $\leqslant x$ and $\#\mathscr{A}$ denotes the number of elements in a set $\mathscr{A}$, while $[o, z]$ denotes the directed segment from o to $z \in \mathbb{C}$.

3. Preliminary Lemmas

The following lemma shows that it suffices to consider the weights $W_\alpha(x)$, $1<\alpha<2$:

Lemma 3.1. *Assume that for each* $\alpha \in (1, 2)$, *there exist even polynomials* $\{S_n(x)\} \equiv \{S_{n,\alpha}(x)\}$ *satisfying* (i), (ii), *and* (iii) *in Theorem* 1.1. *Then, for each* $\alpha > 1$, *there exists even polynomials* $\{S_n(x)\} \equiv \{S_{n,\alpha}(x)\}$ *satisfying* (i), (ii), *and* (iii) *in Theorem* 1.1.

Proof. Let $\alpha > 1$. There exists a non-negative integer m such that $2^m < \alpha \leqslant 2^{m+1}$. Let

$$\beta = \alpha/2^m, \tag{3.1}$$

so that $1 < \beta \leqslant 2$. If $\beta = 2$, then α is a positive even integer, and standard methods [7, 21] show that the partial sums of the entire function $W_\alpha(x)$ (which are even) satisfy (i), (ii), and (iii) of Theorem 1.1. So suppose $1 < \beta < 2$. By hypothesis, there exist even polynomials $S_{n,\beta}(x)$ of degree at most n satisfying

$$S_{n,\beta}(x) \sim W_\beta(x), \qquad |x| \leqslant Cn^{1/\beta}, \tag{3.2}$$

$$|S'_{n,\beta}(x) \leqslant C_1 |x|^{\beta-1} W_\beta(x), \qquad |x| \leqslant Cn^{1/\beta}. \tag{3.3}$$

For each positive integer n, let $k = [n/2^m]$ and let

$$S_{n,\alpha}(x) = S_{k,\beta}(x^{2^m}), \tag{3.4}$$

which is an even polynomial of degree at most n. By (3.1) and (3.2), we see

$$S_{n,\alpha}(x) \sim W_\alpha(x),$$

provided

$$|x^{2^m}| \leqslant Ck^{1/\beta} \sim n^{1/\beta},$$

which is true if $|x| \leqslant C_2 n^{1/\alpha}$. Further, by (3.3) and (3.4) for $|x| \leqslant C_2 n^{1/\alpha}$,

$$\begin{aligned} |S'_{n,\alpha}(x)| &= 2^m |x|^{2^m-1} |S'_{k,\beta}(x^{2^m})| \\ &\leqslant C_1 2^m |x|^{2^m-1} |x|^{2^m(\beta-1)} W_\beta(x^{2^m}) \\ &= C_1 2^m |x|^{\alpha-1} W_\alpha(x), \end{aligned}$$

by (3.1). ∎

The next lemma shows that it suffices to approximate on part of the positive real axis:

LEMMA 3.2. *Let $A>0$. Assume that for each $\beta\in(\frac{1}{2},1)$, there exist polynomials $\{T_n(x)\}\equiv\{T_{n,\beta}(x)\}$ and constants C_1 and C_2 such that*

(i) *$T_n(x)$ has degree at most n;*

(ii) $T_n(x)\sim W_\beta(x)$, $x\in[A, C_1 n^{1/\beta}]$; (3.5)

(iii) $|T_n'(x)|\leqslant C_2|x|^{\beta-1}W_\beta(x)$, $x\in[A, C_1 n^{1/\beta}]$. (3.6)

Then, for each $\alpha>1$, there exist even polynomials $\{S_n(x)\}\equiv\{S_{n,\alpha}(x)\}$ satisfying (i), (ii), *and* (iii) *in Theorem* 1.1.

Proof. By the previous lemma, we need consider only $\alpha\in(1,2)$. Let $\beta=\alpha/2$, so that $\beta\in(\frac{1}{2},1)$. For each positive integer n, let

$$S_{n,\alpha}(x)=T_{[n/2],\beta}(x^2+A),$$

so that $S_{n,\alpha}$ is even and has degree at most n. Since $\alpha=2\beta<2$, we have

$$0\leqslant(x^2+A)^\beta-|x|^\alpha\leqslant C_3,\qquad x\in\mathbb{R},$$

so that by (3.5),

$$S_{n,\alpha}(x)\sim\exp(-(x^2+A)^\beta)\sim\exp(-|x|^\alpha)$$

provided $x^2+A\in[A, C_1[n/2]^{1/\beta}]$, which is true if $|x|\leqslant C_4 n^{1/\alpha}$. Thus, $S_{n,\alpha}$ satisfies (i) and (ii) in Theorem 1.1. Further, by hypothesis, for $|x|\leqslant C_4 n^{1/\alpha}$,

$$\begin{aligned}|S_{n,\alpha}'(x)|&\leqslant 2|x|\,C_2(x^2+A)^{\beta-1}W_\beta(x^2+A)\\&\leqslant\begin{cases}C_5(x^2+A)^{\beta-1/2}W_\alpha(x), & |x|\geqslant 1,\\ C_5|x|\,W_\alpha(x), & |x|\leqslant 1,\end{cases}\\&\leqslant C_6|x|^{\alpha-1}W_\alpha(x),\end{aligned}$$

as $\beta=\alpha/2$ and $\alpha-1<1$. ∎

Various forms of the following lemma are well known in complex function theory, but for completeness we include a full proof of the particular form we need.

LEMMA 3.3. *Let $\phi(t)$ be a function continuous and non-decreasing in $[0,\infty)$, with $\phi(1)=1$ and $\phi(t)\geqslant 0$, $t\in[0,\infty)$. Assume further that*

$$\lim_{t\to\infty}\phi(t)=\infty, \tag{3.7}$$

and assume there exists a non-negative integer l such that

$$\int_1^{\infty} \phi(t)/t^{l+2}\, dt < \infty. \tag{3.8}$$

For $n = 1, 2, 3, \ldots$, let r_n be the smallest positive root of the equation

$$\phi(r_n) = n. \tag{3.9}$$

Let

$$G_\phi(z) = \prod_{n=1}^{\infty} E(-z/r_n, l). \tag{3.10}$$

Then, $G_\phi(z)$ is an entire function, and for a suitable determination of the logarithm,

$$\log G_\phi(z) = (-1)^l H(z) + U(z) + F(z), \qquad z \in \mathbb{C}\backslash(-\infty, 0), \tag{3.11}$$

where $U(z)$ is a polynomial of degree at most l, and

$$H(z) = \int_1^{\infty} \frac{\phi(t)}{t+z}\left(\frac{z}{t}\right)^{l+1} dt, \qquad z \in \mathbb{C}\backslash(-\infty, 0), \tag{3.12}$$

and

$$F(z) = \int_1^{\infty} \frac{[\phi(t)] - \phi(t)}{t+z}\left(\frac{z}{t}\right) dt, \qquad z \in \mathbb{C}\backslash(-\infty, 0). \tag{3.13}$$

Proof. First note that the integral in (3.12) converges uniformly for z in compact subsets of $\mathbb{C}\backslash(-\infty, 0]$ (by (3.8)) and hence defines an analytic function there. Similar remarks apply to the integral in (3.13) as $|[\phi(t)] - \phi(t)| \leqslant 1$. Using the first of the identities,

$$(-1)^l \frac{1}{t+z}\left(\frac{z}{t}\right)^{l+1} = \frac{1}{t+z}\frac{z}{t} + \sum_{j=1}^{l} (-z)^j t^{-j-1} \tag{3.14}$$

$$= \frac{1}{t} - \frac{1}{t+z} + \sum_{j=1}^{l} (-z)^j t^{-j-1}, \tag{3.15}$$

and using (3.12), we see that for $z \in \mathbb{C}\backslash(-\infty, 0)$,

$$(-1)^l H(z) = \int_1^{\infty} \{(\phi(t) - [\phi(t)]) + [\phi(t)]\}(-1)^l \frac{1}{t+z}\left(\frac{z}{t}\right)^{l+1} dt$$

$$= -F(z) - U(z) + (-1)^l \int_1^{\infty} \frac{[\phi(t)]}{t+z}\left(\frac{z}{t}\right)^{l+1} dt, \tag{3.16}$$

by (3.13), and where $U(z)$ is a polynomial of degree at most l. Next, for $n=1, 2,\ldots$, let $\chi_n(t)$ be the characteristic function of the interval $[r_n, \infty)$. Then, for $t\in(0, \infty)$,

$$\sum_{n=1}^{\infty} \chi_n(t) = \#\{r_n : r_n \leqslant t\}$$

$$= \#\{n : n \leqslant \phi(t)\} = [\phi(t)], \tag{3.17}$$

by (3.9) and monotonicity of ϕ. Next, by (3.8) and (3.17),

$$\infty > \int_1^{\infty} [\phi(t)]\, t^{-l-2}\, dt = \sum_{n=1}^{\infty} \int_1^{\infty} \chi_n(t)\, t^{-l-2}\, dt$$

$$= (l+1)^{-1} \sum_{n=1}^{\infty} r_n^{-l-1}. \tag{3.18}$$

Here the interchange of series and integral is justified as all terms in the series and integral(s) are non-negative (Halmos [13, p. 112, Theorem B]). It follows from (3.18) that $G_\phi(z)$ is entire (Boas [2, p. 19]). Now let z be real and positive. Applying (3.17) and noting that all the terms in the series and integral are real and have the same sign, we see that

$$(-1)^l \int_1^{\infty} \frac{[\phi(t)]}{t+z} \left(\frac{z}{t}\right)^{l+1} dt$$

$$= \sum_{n=1}^{\infty} \int_1^{\infty} \frac{\chi_n(t)}{t+z} \left(\frac{z}{t}\right)^{l+1} (-1)^l\, dt$$

$$= \sum_{n=1}^{\infty} \int_{r_n}^{\infty} \left\{ \frac{1}{t} - \frac{1}{t+z} + \sum_{j=1}^{l} (-z)^j t^{-j-1} \right\} dt \qquad \text{(by (3.15))}$$

$$= \sum_{n=1}^{\infty} \left\{ \log(1+z/r_n) + \sum_{j=1}^{l} (-z)^j j^{-1} r_n^{-j} \right\}$$

$$= \sum_{n=1}^{\infty} \log E(-z/r_n, l) = \log G_\phi(z). \tag{3.19}$$

From this last identity and (3.16), we see that (3.11) holds for z real and positive. As both sides of (3.11) are analytic in $\mathbb{C}\backslash(-\infty, 0]$, the result follows. ∎

Our strategy in constructing polynomials satisfying the hypothesis of Lemma 3.2 will be to approximate each of $G_\phi(z)\exp(-U(z))$ and $\exp(-F(z))$ by polynomials along some ray $\{r\exp(i\theta_0) : r\in(0, \infty)\}$, and then to multiply these polynomials as an approximation to

$$\exp((-1)^l H(z)) = G_\phi(z)\exp(-U(z) - F(z)).$$

In the case of Lemma 3.2 we shall choose $\phi(t)=t^{\beta}$, $l=0$, and it will turn out that for suitable $A>0$, $\theta_0 \in (-\pi, \pi)$ and $z=r\exp(i\theta_0)$, $r>0$,

$$|\exp((-1)^l H(z))| \sim \exp(-Ar^{\beta}).$$

4. The Remainder Term $\exp(-F(z))$

The purpose of this section is to prove the following general proposition, which can be used to deal with $\exp(-F(z))$:

Proposition 4.1. *Let $\psi(t)$ be a measurable function on $[1, \infty)$ satisfying*

$$|\psi(t)| \leqslant 1, \qquad t \in [1, \infty), \tag{4.1}$$

and let

$$F_0(z) = \int_1^{\infty} \frac{\psi(t)}{t+z} \frac{z}{t}\, dt, \qquad z \in \mathbb{C}\backslash(-\infty, -1). \tag{4.2}$$

Let $\eta > 1$ and $\{\Gamma_n\}_{n=1}^{\infty}$ be a sequence of positive numbers satisfying

$$1 \leqslant \Gamma_n \leqslant n^2/(\log n)^{2\eta} \qquad \text{for all } n \text{ large enough.} \tag{4.3}$$

Finally, let $-\pi < \theta_0 < \pi$. Then, for each positive integer n, there exist polynomials $P_n(z)$ of degree at most n, such that

$$|P_n(z)\exp(F_0(z))| \sim 1, \tag{4.4}$$

and

$$|P_n'(z)\exp(F_0(z))| \leqslant C(1+|z|)^{-1}, \tag{4.5}$$

for all $z \in [0, \Gamma_n \exp(i\theta_0)]$.

The proof of Proposition 4.1 will be split into several lemmas, all of which assume the hypotheses of Proposition 4.1.

Lemma 4.2. *Consider the ellipse $\mathscr{E}_n$ with foci at 0 and $\Gamma_n \exp(i\theta_0)$, and with major semi-axis equal to $1+\Gamma_n/2$, for each positive integer n* (Fig. 1). *Let $\varepsilon > 0$ be so small that*

$$-\pi+\varepsilon < \theta_0 < \pi - \varepsilon. \tag{4.6}$$

Then there exists $A>0$ such that the set

$$\mathscr{D} = \{re^{i\theta}: r \geqslant A \text{ and } \pi-\varepsilon \leqslant \theta \leqslant \pi+\varepsilon\}, \tag{4.7}$$

does not intersect $\mathscr{E}_n$, $n = 1, 2, 3, \ldots$.

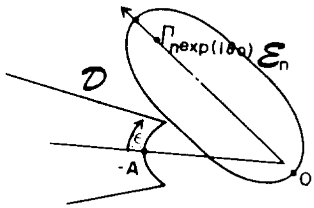

FIGURE 1.

Proof. The point $z = re^{i\theta}$ lies outside $\mathscr{E}_n$ if the sum of its distances to 0 and $\Gamma_n \exp(i\theta_0)$ (the foci of $\mathscr{E}_n$) exceeds $2 + \Gamma_n$ (the major axis of $\mathscr{E}_n$). Now, if $\pi - \varepsilon \leqslant \theta \leqslant \pi + \varepsilon$, then

$$\theta - \theta_0 = (\pi - \theta_0) + (\theta - \pi) \begin{cases} \leqslant \pi - \theta_0 + \varepsilon < 2\pi \\ \geqslant \pi - \theta_0 - \varepsilon > 0, \end{cases}$$

by (4.6). Hence, for some $\eta \in (0, 1)$ independent of θ, $\cos(\theta - \theta_0) \leqslant 1 - \eta < 1$. Then,

$$\begin{aligned} |z| + |z - \Gamma_n \exp(i\theta_0)| &= r + |\Gamma_n - r \exp(i(\theta_0 - \theta))| \\ &\geqslant r + \Gamma_n - r \cos(\theta - \theta_0) \\ &\geqslant \eta r + \Gamma_n > \Gamma_n + 2 \end{aligned}$$

if $r \geqslant 2/\eta$. Thus, we may choose $A = 2/\eta$. ∎

LEMMA 4.3. *Let A and $\mathscr{D}$ be as in Lemma 4.2. Let*

$$F_1(z) = \int_{2A}^{\infty} \frac{\psi(t)}{t+z} \frac{z}{t}\, dt, \qquad z \in \mathbb{C} \backslash (-\infty, -2A). \tag{4.8}$$

Then $F_1(z)$ is analytic in $\mathbb{C} \backslash \mathscr{D}$ and satisfies there

(i) $|F_1'(z)| \leqslant C_1/(|z| + 1)$; (4.9)

(ii) $|F_1(z)| \leqslant C_1 \log(1 + |z|)$. (4.10)

In particular,

(iii) $n^{-C} \leqslant |\exp(-F_1(z))| \leqslant n^C, \qquad z \in \mathscr{E}_n,\ n = 1, 2, 3, \ldots.$ (4.11)

Proof. If $z = re^{i\theta}$ with $|\theta| < \pi$,

$$\begin{aligned} |F_1'(z)| &\leqslant \left| \int_{2A}^{\infty} \frac{\psi(t)}{(z+t)^2}\, dt \right| \\ &\leqslant \int_0^{\infty} \frac{dt}{(t + \eta \cos\theta)^2 + (r \sin\theta)^2} = \frac{\theta}{r \sin\theta}, \end{aligned} \tag{4.12}$$

by (4.1) and (859.163) in Dwight [5, p. 228]. Here, if $\theta = 0$, one must replace $\theta/\sin\theta$ by its limiting value 1. Further, if $|z| \leqslant A$, we see

$$|F_1'(z)| \leqslant \int_{2A}^{\infty} \frac{dt}{(t-A)^2} = A^{-1}. \tag{4.13}$$

From (4.12) and (4.13) we obtain (4.9) for $z \in \mathbb{C}\backslash\mathscr{D}$ and some suitable C_1. Then (4.10) follows as $F_1(0) = 0$. The last assertion of the lemma follows from (4.10) and Lemma 4.2, and as

$$z \in \mathscr{E}_n \Rightarrow |z| \leqslant 1 + \Gamma_n \leqslant 2n^2, \qquad n = 2, 3, \ldots,$$

by (4.3). ∎

LEMMA 4.4. *Let $\mathscr{E}$ be the ellipse with foci at z_1 and z_2 and with semi-axes a and b. Then for any function g, analytic on $\mathscr{E}$ and its interior, one can find polynomials $P_n^*(z)$ of degree at most n, $n = 1, 2, 3, \ldots$, such that*

$$\begin{aligned} &\max\{|g(z) - P_n^*(z)| : z \in [z_1, z_2]\} \\ &\qquad \leqslant 2 \max\{|g(z)| : z \in \mathscr{E}\}\, \rho^{-n}/(\rho - 1), \end{aligned} \tag{4.14}$$

where

$$\rho = 2(a+b)/|z_1 - z_2|. \tag{4.15}$$

Proof. The special case $z_1 = -1$ and $z_2 = 1$ is considered in Lorentz [15, p. 78, inequality (6)]. Since a suitable linear transformation maps $\mathscr{E}$ onto the ellipse with foci at -1, 1 and sum of half-axes ρ, the general case follows. ∎

Proof of Proposition 4.1. It obviously suffices to prove (4.4) and (4.5) for all large enough positive integers n. Let $\mathscr{E}_n$ and $\mathscr{D}$ be as in Lemma 4.2. Since $\mathscr{E}_n$ does not intersect $\mathscr{D}$ (by Lemma 4.2), we can use Lemma 4.4 with $\mathscr{E} = \mathscr{E}_n$ and $g(z) = (d/dz)(\exp(-F_1(z)))$. For this case

$$a = 1 + \Gamma_n/2; \qquad |z_1 - z_2| = \Gamma_n;$$

and

$$b = (a^2 - |z_1 - z_2|^2/4)^{1/2} = (1 + \Gamma_n)^{1/2}.$$

Hence, by (4.15)

$$\rho > 1 + 2\Gamma_n^{-1/2}. \tag{4.16}$$

Applying Lemma 4.4, and using the estimates (4.9), (4.11), and (4.16), we conclude that there exist polynomials $P_n^*(z)$ of degree at most $n-1$, $n=1, 2, 3, \ldots$, such that for $z \in [0, \Gamma_n \exp(i\theta_0)]$,

$$\left|\frac{d}{dz}\exp(-F_1(z)) - P_{n-1}^*(z)\right| \leqslant 2n^C(1+2\Gamma_n^{-1/2})^{-(n-1)\Gamma_n^{-1/2}}$$

$$\leqslant C_3 \exp(-C_4(\log n)^\eta),$$

for some suitable C_3 and C_4, by (4.3). Now define a polynomial $P_n(z)$ of degree at most n, by

$$P_n'(z) = P_{n-1}^*(z) \qquad \text{and} \qquad P_n(0) = 1,$$

to obtain for $z \in [0, \Gamma_n \exp(i\theta_0)]$,

$$\left|\frac{d}{dz}\exp(-F_1(z)) - P_n'(z)\right| \leqslant C_3 \exp(-C_4(\log n)^\eta). \tag{4.17}$$

Since $\exp(-F_1(0)) = 1$, we can integrate to obtain

$$|\exp(-F_1(z)) - P_n(z)| \leqslant C_5 \exp(-C_6(\log n)^\eta), \tag{4.18}$$

for $z \in [0, \Gamma_n \exp(i\theta_0)]$. Using (4.11) and (4.18), we deduce that

$$|P_n(z)\exp(F_1(z))| \sim 1, \qquad z \in [0, \Gamma_n \exp(i\theta_0)].$$

Further, using (4.9), (4.11), and (4.17), we deduce that

$$|P_n'(z)\exp(F_1(z))| \leqslant C_8/(|z|+1), \qquad z \in [0, \Gamma_n \exp(i\theta_0)].$$

To complete the proof of (4.4) and (4.5) if suffices to show that for $z \in [0, \Gamma_n \exp(i\theta_0)]$,

$$|\exp(F_1(z))| \sim |\exp(F_0(z))|.$$

In view of (4.2) and (4.8), this is equivalent to showing

$$\left|\int_1^{2A} \frac{\psi(t)}{t+z}\left(\frac{z}{t}\right) dt\right| \leqslant C_9,$$

for all $z \in [0, \Gamma_n \exp(i\theta_0)]$. For $|z| \geqslant 3A$, such a bound is evident, and for $z \in [0, 3A \exp(i\theta_0)]$, the bound follows by continuity. ∎

5. The Entire Function $G_\phi(z)\exp(-U(z))$

Throughout this section, we assume the hypotheses of Lemma 3.3. First, we establish some properties of G_ϕ and H.

LEMMA 5.1. (i) *There exists* C *such that for* $|z| = r \geqslant 0$,

$$|G_\phi(z)| \leqslant \exp(CH(r)). \tag{5.1}$$

(ii) *For* $z = re^{i\theta}$, $|\theta| < \pi$,

$$|H(z)| \leqslant \sqrt{2}\, H(r) \max\{1, (1 + \cos\theta)^{-1/2}\}. \tag{5.2}$$

(iii) *For* $r > 0$,

$$l \leqslant rH'(r)/H(r) \leqslant l + 1. \tag{5.3}$$

(iv) $H(r)$ *and* $H(r)/r^l$ *are non-decreasing in* $(0, \infty)$.

(v) *For* $r \geqslant 1$,

$$H(1)\, r^l \leqslant H(r) \leqslant H(1)\, r^{l+1}. \tag{5.4}$$

(vi) *For* $z = re^{i\theta}$, $|\theta| < \pi$,

$$|zG'_\phi(z)/G_\phi(z)| \leqslant 2(l+1)\, H(r) \max\{1, (1 + \cos\theta)^{-1}\}. \tag{5.5}$$

Proof. (i) This follows from (2.6.9) in Boas [2, p. 19] and from (3.10) and (3.12) above. Note that $n(t) = [\phi(t)] \leqslant \phi(t)$ in our case.

(ii) From (3.12),

$$|H(z)| \leqslant \int_1^\infty \frac{\phi(t)}{|t+z|} \left(\frac{r}{t}\right)^{l+1} dt. \tag{5.6}$$

Now, if $\cos\theta \geqslant 0$,

$$|t+z|^2 = t^2 + r^2 + 2rt\cos\theta \geqslant t^2 + r^2 \geqslant (t+r)^2/2.$$

On the other hand, if $\cos\theta < 0$, the inequality $2rt \leqslant r^2 + t^2$ yields

$$|t+z|^2 \geqslant (t^2 + r^2)(1 + \cos\theta) \geqslant (t+r)^2(1 + \cos\theta)/2.$$

We deduce that for $|\theta| < \pi$,

$$1/|t+z| \leqslant \sqrt{2} \max\{1, (1 + \cos\theta)^{-1/2}\}/(t+r). \tag{5.7}$$

Together with (5.6) this yields (5.2).

(iii) From (3.12),

$$rH'(r) = \int_1^\infty \frac{\phi(t)}{t+r} \left(\frac{r}{t}\right)^{l+1} \left\{\frac{(l+1)\,t + lr}{t+r}\right\} dt.$$

Then (5.3) follows as

$$l \leqslant \frac{(l+1)\,t+lr}{t+r} \leqslant l+1.$$

(iv) From (5.3), we see that

$$H'(r) \geqslant 0 \qquad \text{and} \qquad \frac{d}{dr}\{H(r)/r^l\} \geqslant 0.$$

(v) This follows by integrating (5.3).

(vi) From (3.19), we see that

$$zG'_\phi(z)/G_\phi(z) = (-1)^l \int_1^\infty \frac{[\phi(t)]}{t+z}\left(\frac{z}{t}\right)^{l+1} \left\{\frac{(l+1)\,t+lz}{t+z}\right\} dt$$

and using (5.7), we easily obtain (5.5). ∎

LEMMA 5.2. *Let*

$$f(z) = G_\phi(z)\exp(-U(z)), \qquad z \in \mathbb{C}. \tag{5.8}$$

Let $R_n(z)$ *be the* $(n+1)$*th partial sum of the Maclaurin series of* $f(z)$, $n = 1, 2, 3, \ldots$. *Let* $\theta_0 \in (-\pi, \pi)$. *Assume that* $\{\xi_n\}$ *is a sequence of positive numbers such that for some* C_1 and C_2,

$$H(\xi_n) \leqslant C_1 n, \qquad n = 1, 2, \ldots, \tag{5.9}$$

and

$$1 \leqslant \xi_n \leqslant C_2 n^{1/l}, \qquad n = 1, 2, \ldots, \text{ if } l > 0, \tag{5.10}$$

$$0 \leqslant \log \xi_n \leqslant C_2 n, \qquad n = 1, 2, \ldots, \text{ if } l = 0. \tag{5.11}$$

Then there exist C_3 *and* C_4 *such that for* $z = r\exp(i\theta_0)$ *and* $r \in (0, C_3, \xi_n]$,

(i) $|R_n(z)| \sim |f(z)|$.

$$\text{(ii)} \quad |R'_n(z)| \leqslant \begin{cases} C_4(H(r)/r)|f(z)|, & r \geqslant 1, \\ C_4(H(r)/r+1)|f(z)|, & r < 1. \end{cases} \tag{5.12}$$

Proof. Let $0 < \varepsilon \leqslant \frac{1}{4}$. Cauchy's integral formula yields for $|z| \leqslant \varepsilon\xi_n$,

$$|R'_n(z) - f'(z)| \leqslant 2(\varepsilon\xi_n)^{-1} \max\{|R_n(t) - f(t)| : |t| \leqslant 2\varepsilon\xi_n\}. \tag{5.13}$$

Further, Cauchy's integral formula yields in the usual way, for $|t| \leqslant 2\varepsilon\xi_n$,

$$\begin{aligned}|R_n(t) - f(t)| &= \left|(2\pi i)^{-1} \int_{|u| = \xi_n} \frac{f(u)}{u-t} \left(\frac{t}{u}\right)^n du\right| \\ &\leqslant 2 \max\{|f(u)| : |u| = \xi_n\}(2\varepsilon)^n \\ &\leqslant 2 \exp(C_5 H(\xi_n) + C_6 \xi_n^l)(2\varepsilon)^n\end{aligned}$$

(by (5.1), (5.8), and as U has degree at most l)

$$\leqslant \exp(C_7 n + n \log(2\varepsilon)), \tag{5.14}$$

by (5.9), (5.10), and (5.11). From (5.13) and (5.14), we deduce

$$|R_n^{(j)}(z) - f^{(j)}(z)| \leqslant \exp(C_8 n + (n-j) \log(2\varepsilon)), \tag{5.15}$$

for $j = 0, 1$, $|z| \leqslant \varepsilon\xi_n$, and some C_8 independent of ε. Next, Lemma 4.3 shows that for $z = r\exp(i\theta_0)$ and $r \in (0, \infty)$,

$$|F(z)| \leqslant C_9 \log(1 + |z|), \tag{5.16}$$

where C_9 is independent of z. Next, by (3.11) and (5.8), for $z \in (0, \xi_n \exp(i\theta_0)]$, we have

$$\begin{aligned}|f(z)| &= |\exp((-1)^l H(z) + F(z))| \\ &\geqslant \exp(-C_{10}\{H(\xi_n) + \log(1 + \xi_n)\})\end{aligned}$$

(by (5.2), monotonicity of H, and (5.16))

$$\geqslant \exp(-C_{11} n), \tag{5.17}$$

by (5.9), (5.10), and (5.11). Together with (5.15), this shows that

$$|R_n(z)| \sim |f(z)|, \qquad z \in (0, \varepsilon\xi_n \exp(i\theta_0)],$$

provided ε is small enough. Next if $z = r\exp(i\theta_0)$ and $r > 0$,

$$\begin{aligned}|f'(z)/f(z)| &\leqslant |G_\phi'(z)/G_\phi(z)| + |U'(z)| \\ &\leqslant \begin{cases} C_{12}(H(r)/r + r^{l-1}), & r \geqslant 1, \\ C_{12}(H(r)/r + 1), & r < 1. \end{cases}\end{aligned} \tag{5.18}$$

Here we have used (5.5) and the fact that U is a polynomial of degree at most l. Using (5.4), (5.15), (5.17), and (5.18), we see that (5.12) follows for $z \in (0, \varepsilon\xi_n \exp(i\theta_0)]$, provided ε is small enough. ∎

6. Proof of Theorem 1.1

To prove Theorem 1.1, it suffices to construct polynomials $\{T_n(x)\}$ satisfying the hypotheses of Lemma 3.2. Let us fix $\frac{1}{2} < \beta < 1$ and let

$$\phi(t) = t^\beta, \qquad t \in [0, \infty).$$

This choice of ϕ satisfies the hypotheses of Lemma 3.3 with $l = 0$. Further, we see from (3.12) that

$$H(z) = \int_0^\infty \frac{t^\beta}{t+z}\left(\frac{z}{t}\right) dt - \int_0^1 \frac{t^\beta}{t+z}\left(\frac{z}{t}\right) dt \tag{6.1}$$

$$= \pi \operatorname{cosec}(\pi\beta)\, z^\beta + O(1), \tag{6.2}$$

uniformly in the sector $|\arg(z)| \leqslant \pi + \varepsilon$ for any $0 < \varepsilon < \pi$. Here we have used (4.1.6) in Boas [2, p. 56] and have applied (5.7) to the second integral in (6.1). Next, as $\frac{1}{2} < \beta < 1$, we can choose $\theta_0 \in (-\pi, \pi)$ such that $\pi/2 < \beta\theta_0 < \pi$, and hence $\cos(\beta\theta_0) < 0$. It then follows that

$$|\exp(H(z))| \sim \exp(-Br^\beta), \qquad z = r\exp(i\theta_0),\ r \in [0, \infty), \tag{6.3}$$

where

$$B = \pi \operatorname{cosec}(\pi\beta) |\cos(\beta\theta_0)| > 0.$$

Now, let us set $\xi_n = \Gamma_n = n^{1/\beta}$, $n = 1, 2, 3, \ldots$. It is clear from (6.2) that (5.9) and (5.11) are satisfied, while as $\beta > \frac{1}{2}$, (4.3) is satisfied. Let $\{R_n\}$ be the polynomials of Lemma 5.2 and $\{P_n\}$ be the polynomials of Proposition 4.1, with $\psi(t) = [\phi(t)] - \phi(t)$, so that $F_0(z) \equiv F(z)$. Further, let

$$V_n(z) = P_{[n/4]}(z)\, R_{[n/4]}(z), \qquad n = 1, 2, 3, \ldots \tag{6.4}$$

and

$$T_n^*(x) = |V_n(x\exp(i\theta_0))|^2, \qquad n = 1, 2, 3, \ldots,\ x \text{ real}, \tag{6.5}$$

so that V_n and T_n^* are polynomials of degree at most $n/2$ and n, respectively. By (6.4), (6.5), (4.4), and Lemma 5.2(i), for $x \in (0, C_1 n^{1/\beta})$ and $z = x\exp(i\theta_0)$,

$$\begin{aligned} T_n^*(x) &\sim |\exp(-F(z))\, G_\phi(z)\exp(-U(z))|^2 \\ &= |\exp(2H(z))| \qquad (\text{by } (3.11)) \\ &\sim \exp(-2Bx^\beta), \end{aligned}$$

by (6.3). If, in addition, $x \geqslant 1$, (4.5) and (5.12) yield

$$\begin{aligned}|T_n^{*\prime}(x)| &= 2|\mathrm{Re}\{V_n'(z)\exp(i\theta_0)\overline{V_n(z)}\}| \\ &\leqslant C_3 \exp(-2Bx^{\beta})\{x^{-1} + H(x)/x\} \\ &\leqslant C_4 \exp(-2Bx^{\beta})\, x^{\beta-1},\end{aligned}$$

by (6.2). Finally, let

$$T_n(x) = T_n^*(x(2B)^{-1/\beta}), \qquad n = 1, 2, \ldots. \quad \blacksquare$$

7. Proof of Corollaries 1.2 and 1.3

We shall prove both Corollaries 1.2 and 1.3 in a more general form, but need some preliminary lemmas:

Lemma 7.1. *Let $0 < p \leqslant \infty$. Let $0 < \varepsilon < 1$. There exists C depending on p and ε only, such that for $n = 1, 2, 3, \ldots$ and all polynomials P of degree at most n,*

$$\|P'\|_{L_p[-\varepsilon,\varepsilon]} \leqslant Cn\|P\|_{L_p[-1,1]}.$$

Proof. For $1 \leqslant p \leqslant \infty$, this follows from Theorems 9.16 and 9.19 in Nevai [24, pp. 163–164]. For $0 < p < 1$, this follows from Theorem 5 in Nevai [23, p. 243]. $\blacksquare$

Lemma 7.2. *Let $\alpha > 0$. Let $0 < p \leqslant \infty$. There exist C_1 and C_2 depending on α and p only, such that for all polynomials P of degree at most n, $n = 1, 2, 3, \ldots$,*

$$\|PW_\alpha\|_{L_p(\mathbb{R})} \leqslant C_1 \|PW_\alpha\|_{L_p(-C_2 n^{1/\alpha}, C_2 n^{1/\alpha})}.$$

Proof. See Lubinsky [16, Theorem A] or Mhaskar and Saff [20, Lemma 6.3]. $\blacksquare$

We can now prove

Theorem 7.3. (Local Markov–Bernstein Inequality). *Let $\alpha > 1$. Let $0 < p \leqslant \infty$. Let $0 < \eta < \xi < \infty$. There exist $C = C(\eta, \xi, \alpha, p)$ only, such that for all polynomials P of degree at most n, $n = 1, 2, 3, \ldots$,*

$$\|P'W_\alpha\|_{L_p(-\eta n^{1/\alpha}, \eta n^{1/\alpha})} \leqslant Cn^{1-1/\alpha}\|PW_\alpha\|_{L_p(-\xi n^{1/\alpha}, \xi n^{1/\alpha})}. \tag{7.1}$$

Proof. Let $\mathscr{I}_n = (-\eta n^{1/\alpha}, \eta n^{1/\alpha})$ and $\mathscr{J}_n = (-\xi n^{1/\alpha}, \xi n^{1/\alpha})$, $n = 1, 2, 3, \ldots$. By Theorem 1.1, there exists a positive integer J, independent of n, and polynomials $S_n(x)$ of degree at most Jn, $n = 1, 2, \ldots$, such that

$$S_n(x) \sim W_\alpha(x), \qquad |x| \leqslant \xi n^{1/\alpha}, \tag{7.2}$$

and

$$|S_n'(x)| \leqslant Cn^{1-1/\alpha} W_\alpha(x), \qquad |x| \leqslant \xi n^{1/\alpha}. \tag{7.3}$$

Then, for each polynomial $P(x)$ of degree at most n,

$$\|P' W_\alpha\|_{L_p(\mathscr{I}_n)} \leqslant C_1 \|P' S_n\|_{L_p(\mathscr{I}_n)}$$

(by (7.2))

$$\begin{aligned} &= C_1 \|(PS_n)' - PS_n'\|_{L_p(\mathscr{I}_n)} \\ &\leqslant C_2 \{ \|(PS_n)'\|_{L_p(\mathscr{I}_n)} + n^{1-1/\alpha} \|PW_\alpha\|_{L_p(\mathscr{I}_n)} \}, \end{aligned} \tag{7.4}$$

by (7.3). This last step is valid even if $0 < p < 1$, provided C_2 is large enough. Since PS_n is a polynomial of degree at most $(J+1)\,n$, we can apply Lemma 7.1 after transforming the interval $\mathscr{I}_n$ to $(-\varepsilon, \varepsilon)$ and $\mathscr{J}_n$ to $(-1, 1)$, with $\varepsilon = \eta/\xi$, to deduce

$$\begin{aligned} \|(PS_n)'\|_{L_p(\mathscr{I}_n)} &\leqslant C_3 n^{1-1/\alpha} \|PS_n\|_{L_p(\mathscr{J}_n)} \\ &\leqslant C_4 n^{1-1/\alpha} \|PW_\alpha\|_{L_p(\mathscr{J}_n)}, \end{aligned}$$

by (7.2). Together with (7.4), this yields (7.1). ∎

We note that essentially the above idea appears in Bonan [3] and Bonan and Nevai [4], but the method was rediscovered by the present authors. It is clear that Corollary 1.3 follows from Lemma 7.2 and Theorem 7.3. The following result is new only for $1 < \alpha < 2$:

THEOREM 7.4. *Let $0 < p \leqslant \infty$, $\alpha > 1$, and j be a non-negative integer. Then there exist C_1 and C_2 depending only on j, p, and α such that*

$$\text{(i)} \quad \lambda_{n,p}(W_\alpha, j, x) \sim (n^{1/\alpha - 1})^{j+1/p} W_\alpha(x), \qquad |x| \leqslant C_1 n^{1/\alpha}; \tag{7.5}$$

$$\text{(ii)} \quad \lambda_{n,p}(W_\alpha, j, x) \geqslant C_2 (n^{1/\alpha - 1})^{j+1/p} W_\alpha(x), \qquad x \in \mathbb{R}. \tag{7.6}$$

Proof. It suffices to prove (7.6), since matching upper bounds for $\lambda_{n,p}(W_\alpha, j, x)$ for $|x| \leqslant C_1 n^{1/\alpha}$ appear in [17, Theorem 3.6(ii)]. We start by considering the case $j = 0$. Let $0 < \eta < \xi < \infty$, and $\{S_n\}$ be the polynomials of degree at most Jn satisfying (7.2) and (7.3). Then, for $|x| \leqslant \eta n^{1/\alpha}$, (2.1) and (7.2) show that

$$\begin{aligned} \lambda_{n,p}(W_\alpha, j, x)/W_\alpha(x) &\geqslant C \inf_{\mathscr{P}_{n-1}} \|PS_n\|_{L_p(-\xi n^{1/\alpha}, \xi n^{1/\alpha})}/|PS_n|(x) \\ &\geqslant C \inf_{\mathscr{P}_{(J+1)n-1}} \|P\|_{L_p(-\xi n^{1/\alpha}, \xi n^{1/\alpha})}/|P(x)| \\ &\geqslant C_1 (n^{1/\alpha - 1})^{1/p}, \end{aligned} \tag{7.7}$$

by Lemma 6.3.5 and Theorem 6.3.13 [24, pp. 108, 113] and by transforming the interval $(-\xi n^{1/\alpha}, \xi n^{1/\alpha})$ to $(-1, 1)$. Although Nevai's result is proved for $0 < p < \infty$, the last step is trivial if $p = \infty$. This establishes (7.6) for $j = 0$ and $|x| \leqslant \eta n^{1/\alpha}$, $\eta > 0$ arbitrary. Now, by Lemma 7.2, we can choose $\eta > 0$ such that for all polynomials P of degree at most n,

$$\begin{aligned} \|PW_\alpha\|_{L_\infty(\mathbb{R})} &\leqslant C\|PW_\alpha\|_{L_\infty(-\eta n^{1/\alpha}, \eta n^{1/\alpha})} \\ &\leqslant C_2\|PW_\alpha\|_{L_p(\mathbb{R})}(n^{1-1/\alpha})^{1/p}, \end{aligned} \tag{7.8}$$

by (7.7) and the definition (2.1) of $\lambda_{n,p}(W_\alpha, 0, x)$. Hence, (7.6) holds for $j = 0$ and all $x \in \mathbb{R}$. This last trick is essentially due to Mhaskar and Saff [20, Theorem 6.5].

To prove (7.6) for $j = 1, 2, 3, \ldots$, we note that for any polynomial P of degree at most n, Corollary 1.3 yields

$$\begin{aligned} \|P^{(j)}W_\alpha\|_{L_\infty(\mathbb{R})} &\leqslant C_3(n^{1-1/\alpha})^j\|PW_\alpha\|_{L_\infty(\mathbb{R})} \\ &\leqslant C_4(n^{1-1/\alpha})^{j+1/p}\|PW_\alpha\|_{L_p(\mathbb{R})}, \end{aligned}$$

by (7.8). This establishes (7.6) in the general casse. ∎

We can now prove that assertions (i) and (ii) of Theorem 1.1 cannot hold for any $\alpha \in (0, 1]$. For if they did, the method of proof of Theorem 7.4 would show that (7.6) holds for some $\alpha \in (0, 1]$. If $\alpha < 1$, this would contradict the fact that $\lambda_{n,p}$ is non-increasing in n. If $\alpha = 1$, then (7.6) and (2.3) would imply that

$$\lambda_n(W_1^2, x) \geqslant CW_1^2(x), \qquad x \in \mathbb{R},$$

which would contradict the known upper bounds for the Christoffel functions [12] as well as the fact that the Hamburger moment problem for W_1^2 is determinate.

COROLLARY 7.5. *Let $\alpha > 1$ and j be a non-negative integer. Then, for $n = j+1, j+2, \ldots$,*

$$\sum_{k=0}^{n-1} \{p_k^{(j)}(W_\alpha^2; x)\}^2 W_\alpha^2(x) \begin{cases} \sim (n^{1-1/\alpha})^{2j+1}, & |x| \leqslant C_1 n^{1/\alpha}, \\ \leqslant C_2(n^{1-1/\alpha})^{2j+1}, & x \in \mathbb{R}. \end{cases}$$

Proof. This follows from (2.2) and Theorem 7.4. ∎

The following weighted Nikolskii inequality appears in Mhaskar and Saff [20, Theorem 3.1] with an extra factor of $\log n$ for $1 < \alpha < 2$.

THEOREM 7.6. *Suppose $\alpha > 1$ and $0 < p < r \leqslant \infty$. Then*

$$\|PW_\alpha\|_{L_r(\mathbb{R})} \leqslant C(n^{1-1/\alpha})^{1/p-1/r}\|PW_\alpha\|_{L_p(\mathbb{R})}, \tag{7.9}$$

for all polynomials P of degree at most n, and with $C = C(\alpha, p, r)$ only.

Proof. For $r = \infty$, we proved (7.9) in Theorem 7.4—see (7.8). Suppose now $0 < p < r < \infty$. Then

$$\begin{aligned}\|PW_\alpha\|^r_{L_r(\mathbb{R})} &= \int_{-\infty}^{\infty} |PW_\alpha(x)|^{r-p}|PW_\alpha(x)|^p\,dx \\ &\leqslant \|PW_\alpha\|^{r-p}_{L_\infty(\mathbb{R})}\|PW_\alpha\|^p_{L_p(\mathbb{R})} \\ &\leqslant C(n^{1-1/\alpha})^{(1/r)(r-p)}\|PW_\alpha\|^{r-p}_{L_r(\mathbb{R})}\|PW_\alpha\|^p_{L_p(\mathbb{R})},\end{aligned}$$

by what has already been proved. Then (7.9) follows on taking pth roots. ∎

The next result is new only for $1 < \alpha < 2$:

THEOREM 7.7. *Let $\alpha > 1$. There exists C such that for every pair of consecutive zeros x_{jn} and $x_{j+1,n}$ of $p_n(W^2_\alpha; x)$ lying in $(-Cn^{1/\alpha}, Cn^{1/\alpha})$, we have*

$$x_{jn} - x_{j+1,n} \sim n^{-1+1/\alpha}.$$

Proof. The requisite lower bounds for $x_{jn} - x_{j+1,n}$ may be proved as in Theorem 5.1 in Freud [10, p. 36], using Corollary 7.5. The upper bounds were proved in Freud [9, p. 294]. ∎

Finally, we note that Nevai [22, p. 336] showed that the leading coefficient $\gamma_n = \gamma_n(W^2_\alpha)$ of $p_n(W^2_\alpha; x)$ satisfies $\gamma_{n-1}/\gamma_n \sim n^{1/\alpha}$, $n = 1, 2, ..., \alpha \geqslant 1$.

Together with Corollaries 1.2 and 1.3 and Theorem 7.7, this may be used to extend the approximation theoretic results in [10, 11] and possibly those discussed in [19] to the weights W_α, $1 < \alpha < 2$.

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REFERENCES

1. F. F. ABI-KHUZAM, The asymptotic behaviour of a Lindelöf function and its coefficients, *J. Math. Anal. Appl.* **93** (1983), 495–526.
2. R. BOAS, "Entire Functions," Academic Press, New York, 1954.

3. S. BONAN, "Weighted Mean Convergence of Lagrange Interpolation," Ph.D. thesis, Ohio State University, Columbus, 1982.
4. S. BONAN AND P. NEVAI, A Markov–Bernstein theorem, manuscript.
5. H. B. DWIGHT, "Tables of Integrals and Other Mathematical Data," 4th ed., Macmillan Co., New York, 1961.
6. G. FREUD, On polynomial approximation with the weight $\exp(-x^{2k}/2)$, *Acta Math. Hungar.* **24** (1973), 363–371.
7. G. FREUD, On the converse theorems of weighted polynomial approximation, *Acta. Math. Hungar.* **24** (1973), 389–397.
8. G. FREUD, "On Polynomial Approximation with Respect to General Weights," Lecture Notes in Mathematics, Vol. 399 (H. E. Garnir, K. R. Unni, and J. H. Williamson, Eds.), pp. 149–179, Springer-Verlag, Berlin/New York, 1974.
9. G. FREUD, On the theory of one sided weighted polynomial approximation, *in* "Approximation Theory and Functional Analysis" (P. L. Butzer, J. P. Kahane, and B. Sz–Nagy, Eds.), pp. 285–303, Birkhäuser, Basel, 1974.
10. G. FREUD, On Markov–Bernstein type inequalities and their applications, *J. Approx. Theory* **19** (1977), 22–37.
11. G. FREUD, Markov–Bernstein type inequalities in $L_p(-\infty, \infty)$, *in* "Approximation Theory II" (G. G. Lorentz, C. K. Chui, and L. L. Schumaker, Eds.), pp. 369–377, Academic Press, New York, 1976.
12. G. FREUD, A. GIROUX, AND Q. I. RAHMAN, Sur l'Approximation Polynomiale Avec Poids $\exp(-|x|)$, *Canad. J. Math.* **30** (1978), 358–372.
13. P. R. HALMOS, "Measure Theory," Springer-Verlag, New York/Berlin, 1974.
14. B. JA. LEVIN, "Distribution of Zeros of Entire Functions," Translations of Mathematical Monographs, Vol. 5, Amer. Math. Soc., Providence, R.I., 1964.
15. G. G. LORENTZ, "Approximation of Functions," Holt, Rinehart & Winston, New York, 1966.
16. D. S. LUBINSKY, A weighted polynomial inequality, *Proc. Amer. Math. Soc.* **92** (1984), 263–267.
17. D. S. LUBINSKY, Estimates of Freud–Christoffel functions for some weights with the whole real line as support, *J. Approx. Theory* **44** (1985), 343–379.
18. D. S. LUBINSKY, Gaussian quadrature, weights on the whole real line and even entire functions with non-negative even order derivatives, *J. Approx. Theory* **46** (1986), 297–313.
19. H. N. MHASKAR, Weighted polynomial approximation, *J. Approx. Theory* **46** (1986), 100–110.
20. H. N. MHASKAR AND E. B. SAFF, Extremal problems for polynomials with exponential Weights, Trans. *Amer. Math. Soc.* **285** (1984), 203–234.
21. P. NEVAI, Some properties of polynomials orthogonal to the weight $(1+x^{2k})^\alpha \exp(-x^{2k})$ and their applications to approximation theory, *Soviet Math. Dokl.* **14** (1973), 1116–1119.
22. P. NEVAI, Polynomials orthogonal on the real axis to the weight $|x|^\alpha \exp(-|x|^\beta)$, I, *Acta. Math. Hungar.* **24** (1973), 335–342. [Russian]
23. P. NEVAI, Bernstein's inequality in L^p for $0 < p < 1$, *J. Approx. Theory* **27** (1979), 239–243.
24. P. NEVAI, Orthogonal polynomials, *Mem. Amer. Math. Soc.* **18**, No. 213 (1979).
25. P. NEVAI, Orthogonal polynomials on infinite intervals, *in* "Proceedings, International Conference on Special Functions, Torino, Italy, 1984," in press.